Recall! geometric series

$$\sum_{n=1}^{\infty} x^n$$ converges

absolutely for $x \in (-1,1)$.

Back to Power Series

Theorem: (radius of convergence)

Given $\sum_{n=0}^{\infty} a_n x^n$, suppose

the series converges at x_0.

Then the series converges

for all x, $|x| < |x_0|$.

proof:

$$|a_n x^n|$$

$$= |a_n| \, |x|^n$$

$$= |a_n| \, |x_0|^n \left| \frac{x}{x_0} \right|^n$$

$\sum_{n=0}^{\infty} a_n x_0^n$ converges,

so $\lim_{n\to\infty} a_n x_0^n = 0$.

Since $(a_n x_0^n)_{n=0}^{\infty}$ converges, it is bounded.

Therefore, $\exists\ M \geq 0$ with

$$|a_n x_0^n| \leq M \quad \forall\ n \in \mathbb{N} \cup \{0\}.$$

Hence,

$$\begin{aligned} |a_n x^n| &\leq |a_n| \, |x_0|^n \left| \frac{x}{x_0} \right|^n \\ &\leq M \left| \frac{x}{x_0} \right|^n \end{aligned}$$

Since $|x| < |x_0|$,

$$M \sum_{n=0}^{\infty} \left| \frac{x}{x_0} \right|^n$$ is convergent,

and so by comparison,

$$\sum_{n=0}^{\infty} |a_n x^n|$$ converges. $\square$

Definition: (radius of convergence)

Given $\sum_{n=0}^{\infty} a_n x^n$, let

$$R = \sup\left\{ x_0 \geq 0 \;\middle|\; \sum_{n=0}^{\infty} a_n x_0^n \text{ converges} \right\}.$$

This is called the radius of convergence for $\sum_{n=0}^{\infty} a_n x^n$. It may be the case that $R = 0$ or $R = \infty$ (sup does not exist).

Remarks

1) Changing the center –

look at

$$\sup \left\{ x_0 - c \;\middle|\; x_0 - c \geq 0 \text{ and } \sum_{n=0}^{\infty} a_n (x_0 - c)^n \text{ converges} \right\}$$

This will be R.

2) How do you find it?

Ratio test!

Example 1:

Find the radius of convergence for

$$\sum_{n=0}^{\infty} \frac{(nx)^n}{n!}$$

Use ratio test! the series converges absolutely when $\lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| < 1$.

$$b_n = \frac{(nx)^n}{n!}$$

$$b_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$$

$$\frac{b_{n+1}}{b_n} = \cancel{(n+1)} \times \left(\frac{n+1}{n}\right)^n \cdot \frac{1}{\cancel{n+1}}$$

$$\left| \frac{b_{n+1}}{b_n} \right| = |x| \left(\frac{n+1}{n}\right)^n$$

$$\to |x| e$$

as $n \to \infty$

We need

$e|x| < 1$, so

the radius of

convergence is $\frac{1}{e}$.

Diverges for all x, $|x| > \frac{1}{e}$.

What about $x = \pm\frac{1}{e}$?

We don't know!

Endpoints and Interval of Convergence

We know from previous theorem that if $\sum_{n=0}^{\infty} a_n x_0^n$ converges, then so does $\sum_{n=0}^{\infty} a_n x^n$ $\forall$ x, $|x| < |x_0|$. What about equality?

Lemma: (Abel)

Suppose $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences of real numbers, $\exists\ M \geq 0$ with $\left|\sum_{n=1}^{k} a_n\right| \leq M \ \forall\ k \geq 1$ and $(b_n)_{n=1}^{\infty}$ is non-negative and decreasing. Then $\forall\ m \geq 0$,

$$\boxed{\left|\sum_{n=m+1}^{k} a_n b_n\right| \leq 2Mb_1.}$$

<u>proof</u>: Let $a_0 = 0$ and

$$S_k = \sum_{n=0}^{k} a_n \quad (S_0 = 0).$$

$$S_k - S_{k-1} = a_k \quad \forall k \geq 1.$$

$$\sum_{n=m+1}^{k} a_n b_n = \sum_{n=m+1}^{k} (S_n - S_{n-1}) b_n$$

$$= \sum_{n=m+1}^{k} S_n b_n - \sum_{n=m+1}^{k} S_{n-1} b_n$$

Consider $\sum_{n=m+1}^{k} S_n b_n - \sum_{n=m+1}^{k} S_{n-1} b_n$.

$t = n-1$

Re-index the second sum as

$$\sum_{n=m}^{k-1} S_n b_{n+1}.$$ Add again

$$\sum_{n=m+1}^{k-1} S_n (b_n - b_{n+1}) - S_m b_{m+1}$$

$$+ S_k b_k = \sum_{n=m+1}^{k} a_n b_n$$

Then

$$\left|\sum_{n=m+1}^{k} a_n b_n\right|$$

$$\leq \sum_{n=m+1}^{k-1} |S_n| (b_n - b_{n+1})$$

$$+ |S_m| b_{m+1} + |S_k| b_k$$

$$\leq \sum_{n=m+1}^{k-1} M (b_n - b_{n+1}) + M b_{m+1} + M b_k$$

$$= M \sum_{n=m+1}^{k-1} (b_n - b_{n+1}) = M b_{m+1} - M b_k$$

So we get

$$\left| \sum_{n=m+1}^{k} a_n b_n \right| \le 2M b_{m+1}$$

$$\le 2M b_1 \quad \square$$

Theorem (Abel)

Suppose $\sum_{n=0}^{\infty} a_n x^n$ converges

at $x = R \geq 0$. Then

the series converges

uniformly on $[0, R]$.

proof: Show that the sequence of partial sums is uniformly Cauchy.

Let S_k denote k^{th} partial sum

$$S_k = \sum_{n=0}^{k} a_n x^n.$$

Consider $(k \geq m)$

$$|S_k - S_m|$$

$$= \left| \sum_{n=m+1}^{k} a_n x^n \right|$$

$$= \left| \sum_{n=m+1}^{k} a_n R^n \left(\frac{x}{R}\right)^n \right|$$

$\sum_{n=0}^{\infty} a_n R^n$ converges $\Rightarrow$

$$\Rightarrow \quad T_k = \sum_{n=0}^{k} a_n R^n$$

is Cauchy. Choose

N so that $\forall\ m, k \geq N$,

$$|T_k - T_m| < \frac{\varepsilon}{2}.$$

Then using the previous lemma, $\forall\ m, k \geq N$,

$$\left|\sum_{n=m+1}^{k} a_n x^n\right|$$

$$= \left|\sum_{n=m+1}^{k} a_n R^n \left(\frac{x}{R}\right)^n\right|$$

$$\le 2\left(\frac{\varepsilon}{2}\right)\left(\frac{x}{R}\right)$$

$$< 2\left(\frac{\varepsilon}{2}\right) = \varepsilon. \qquad \square$$